MATH 2230 A

Tutorial 8: 2-11-2016

Topics: ∘ Cauchy integral formula and related example

Last time: - Contour integral of rational function

$$R(x) = \frac{p(x)}{q(x)}$$

- $\displaystyle\int_{C(i)} \frac{z^3-2z^2+3z-2}{z^2+1}\,dz$, $C(i) = \{z \in \mathbb{C} \mid |z-i| = 1\}$

$$\overset{\text{Long Division}}{=} \int_{C(i)} \left(z-2+\frac{2z}{z^2+1}\right) dz$$

$$\overset{\text{Partial Fraction}}{=} \int_{C(i)} \left(z-2+\frac{1}{z-i}+\frac{1}{z+i}\right) dz$$

$$= 2\pi i$$

Q: Can we apply this idea to other analytic function $f(z)$?

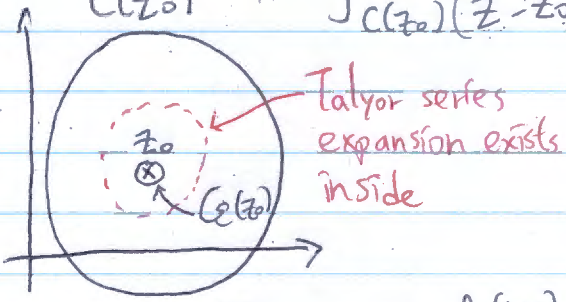
Consider the integral $\displaystyle\int_{C(z_0)} \frac{f(z)}{(z-z_0)^n}\,dz$, where f is analytic in a domain containing $C(z_0)$.

☆ If we assume that $f(z)$ has a Talyor series expansion in a small neighbourhood of z_0, then we have

$$f(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n, \text{ where } C_n = \frac{f^{(n)}(z_0)}{n!}$$

$$= C_0 + C_1(z-z_0) + \dots + C_{n-1}(z-z_0)^{n-1} + C_n(z-z_0)^n + C_{n+1}(z-z_0)^{n+1} + \dots$$

$$\Rightarrow \frac{f(z)}{(z-z_0)^n} = \underbrace{\frac{C_0}{(z-z_0)^n} + \frac{C_1}{(z-z_0)^{n-1}} + \dots}_{\text{Anti-derivative exists}} + \frac{C_{n-1}}{z-z_0} + \underbrace{C_n + C_{n+1}(z-z_0) + \dots}_{\text{analytic}}$$



$$\Rightarrow \int_{C(z_0)} \frac{f(z)}{(z-z_0)^n}dz = \int_{C_\varepsilon(z_0)} \frac{f(z)}{(z-z_0)^n}dz$$

$$= \int_{C_\varepsilon(z_0)} \frac{c_{n-1}}{z-z_0}dz$$

$$= 2\pi i\, c_{n-1}$$

$$\Rightarrow \quad f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\int_{C(z_0)} \frac{f(z)}{(z-z_0)^n}dz$$

Cauchy Integral formula:

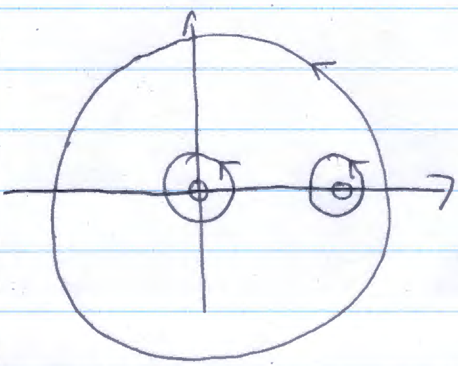
(1) $$f(z_0) = \frac{1}{2\pi i}\int_{C(z_0)} \frac{f(z)}{z-z_0}dz$$

(2) $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Example: 1) Compute the integral

$$I = \int_{C_8(0)} \frac{e^z}{z(z-7)}dz$$

Ans: Note that



$$I = \int_{C_8(0)} \frac{e^z}{z(z-7)}dz$$

$$= \int_{C(0)} \frac{e^z/z-7}{z}dz + \int_{C(7)} \frac{e^z/z}{z-7}dz$$

$$= 2\pi i\left(e^0/(0-7) + e^7/7 \right)$$

$$= (2\pi i)\frac{e^7-1}{7}$$ //

2) Compute the integral

$$I = \int_{C_3(2i)} \frac{\sinh z}{(z-\pi i)^{n+1}}dz \quad \text{, where } n \in \mathbb{Z}$$

Ans: Case I: If $n \leq -1$, then the function $\sinh z\,(z-\pi i)^{-n-1}$ is analytic inside $C_3(2i)$. So $I = 0$.

Case II: If $n > -1$, by Cauchy Integral Formula,

$$I = \int_{C_3(2i)} \frac{\sinh z}{(z-\pi i)^{n+1}}\,dz$$

$$= \frac{2\pi i}{n!} \cdot \sinh^{(n)}(\pi i)$$

$$= \begin{cases} \frac{2\pi i}{(2k)!} \sinh(\pi i) & \text{when } n = 2k \\ \frac{2\pi i}{(2k+1)!} \cosh(\pi i) & \text{when } n = 2k+1 \end{cases}$$

$$= \begin{cases} 0 & \text{when } n = 2k \\ -\frac{2\pi i}{(2k+1)!} & \text{when } n = 2k+1 \end{cases}$$

3) Compute $\int_0^{2\pi} \sin^{2n} t\,dt$, $n \in \mathbb{N}$.

Ans: $$\int_0^{2\pi} \sin^{2n} t\,dt = \int_0^{2\pi} \left(\frac{e^{it}-e^{-it}}{2i}\right)^{2n} dt$$

$$= \frac{1}{(-1)^n 4^n} \int_0^{2\pi} (e^{it}-e^{-it})^{2n}\,dt$$

$$= \frac{1}{(-1)^n 4^n} \int_0^{2\pi} \left(\frac{e^{2it}-1}{e^{it}}\right)^{2n} dt$$

$$= \frac{1}{(-1)^n 4^n} \int_0^{2\pi} \frac{(e^{2it}-1)^{2n}}{e^{2nit}}\,dt$$

$$= \frac{1}{i(-1)^n 4^n} \int_{C(0)} \frac{(z^2-1)^{2n}}{z^{2n+1}}\,dz$$

Since $\displaystyle\int_{C(0)} \frac{(z^2-1)^{2n}}{z^{2n+1}}\,dz = \int_0^{2\pi} \frac{(e^{2it}-1)^{2n}}{e^{(2n+1)it}} \cdot ie^{it}\,dt = i\int_0^{2\pi} \frac{(e^{2it}-1)^{2n}}{e^{2nit}}\,dt$

Let $f(z) = (z^2-1)^{2n}$.

We want to find $f^{(2n)}(0)$.

By Binomial formula, $f(z) = \sum_{k=0}^{2n} \binom{2n}{k} (z^2)^k (-1)^{2n-k}$

$$= \sum_{k=0}^{2n} \binom{2n}{k} (z^{2k}) (-1)^{2n-k}$$

Hence, $f^{(2n)}(0) = (2n)! \binom{2n}{n} (-1)^n$

$$\therefore \int_0^{2\pi} \sin^{2n} t \, dt = \frac{1}{i(-1)^n 4^n} \int_{C(0)} \frac{(z^2-1)^{2n}}{z^{2n+1}} dz$$

$$= \frac{1}{i(-1)^n 4^n} \cdot \frac{2\pi i}{(2n)!} f^{(2n)}(0)$$

$$= \frac{1}{i(-1)^n 4^n} \cdot \frac{2\pi i}{(2n)!} (2n)! \binom{2n}{n} (-1)^n$$

$$= \frac{2\pi}{4^n} \binom{2n}{n}$$

Ex: Try to find $\int_0^{2\pi} \cos^{2n} t \, dt$.

4) Let $n \in \mathbb{N}$.

Suppose f is an entire function such that

$|f(z)| \leq M |z|^n$ when $|z|$ is sufficiently large, where $M > 0$ is a constant.

Show that $f(z)$ must be a polynomial of degree at most n.

Ans: Let $R > 0$ be sufficiently large, $z_0 \in \mathbb{C}$.

By CIF,

$$|f^{(n+1)}(z_0)| = \left| \frac{(n+1)!}{2\pi i} \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+2}} dz \right|$$

Note that on $C_R(z_0)$, $z = z_0 + Re^{i\theta}$, $\theta \in [0, 2\pi]$.

(1) $|f(z)| \le M|z_0 + Re^{i\theta}|^n \le M(|z_0| + R)^n$

(2) $|z - z_0|^{n+2} = |Re^{i\theta}|^{n+2} = R^{n+2}$

$$\therefore \ |f^{(n+1)}(z_0)| \le \frac{(n+1)!}{2\pi} \cdot \frac{M(|z_0|+R)^n}{R^{n+2}} \cdot 2\pi R$$

$$\to 0 \quad \text{as } R \to \infty.$$

$\therefore \ f^{(n+1)}(z_0) = 0 \quad \forall z_0 \in \mathbb{C}.$

$$\therefore \begin{cases} f^{(n)}(z_0) = a_n & \text{for some } a_n \in \mathbb{C} \\ f^{(n-1)}(z_0) = a_n z + a_{n-1} & \text{for some } a_{n-1} \in \mathbb{C} \\ \quad \vdots & \\ f(z_0) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 & \end{cases}$$

for some $a_i \in \mathbb{C}$.